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# Hyper-Charged Vortices and Strings with Signature Change Horizon

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## Abstract

We show that self-dual Nielsen Olesen (NO) vortices in 3 dimensions give rise to a class of exact solutions when coupled to Einstein Maxwell Dilaton gravity obeying the Majumdar-Papapetrou(MP) relation between gravitational and Maxwell couplings , provided certain Chern-Simons type interactions are present. The metric may be solved for explicitly in terms of the NO vortex function and becomes degenerate at scales $r_H \sim l_S exp(\frac{l_S}{l_P})$ where $l_S$ is the vortex core size and $l_P$ the Planck length. For typical $l_S \geq 10^4 l_P$ the horizon is thus pushed out to exponentially large scales. In the intermediate asymptotic region (IAR) $l_S << r << r_H$ there is a logarithmic deviation of the metric from the flat metric and of the electric field from that of a point charge (which makes it decrease *slower* than $r^{-1}$ :hence the prefix hyper). In the IAR the ADM energy and charge integrals increase logarithmically with the distance from the core region and finally diverge at the signature change horizon. String solutions in $4+p$ dimensions are obtained by replacing the Maxwell field by an antisymmetric tensor field (of rank $2+p$) and have essentially similar properties with $r_H \sim l_S exp((\frac{l_S}{l_P})^{2+p})$ and with the antisymmetric charge playing the role of the topological electric charge .

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In an astonishing recent paper [1] it was shown that the self dual "instantonic" solitons of $4+1$ dimensional Yang-Mills (YM) theory and the self dual BPS [2] monopoles of $3+1$ dimensional Yang-Mills-Higgs theory give rise to a class of exact stable static solutions even after they are coupled to Einstein-Maxwell(EM) gravity at the Majumdar- Papapetrou(MP) point[3] (i.e the Maxwell coupling is fixed by the gravitational coupling so that the static Newton and Couloumb potentials are equal in magnitude). Important additional requirements are the inclusion of certain non-minimal terms of form $\epsilon^{MNLPQ} A_M F^A_{NL} F^A_{PQ}$ in $4+1$ dimensions (and its dimensional reduction in $3+1$ dimensions) and coupling to a dilaton in $3+1$ dimensions. Here $A_M, A^A_M$ are the Electromagnetic and YM gauge potentials and $F^A_{MN}$ is the YM field strength . These terms are responsible for the soliton acquiring a topological electric charge coupled to the Maxwell field by a mechanism similar to one proposed for $4+1$ Yang Mills Chern Simons(YMCS) theory solitons in [4, 5, 6]. This electric charge acts as a Bogomol'nyi bound under the ADM [7] mass of field configurations in the theory. The self dual solitonic solutions of Ref.[1] saturate this bound in such a way as to imply the existence of a Killing spinor with respect to a certain Einstein Maxwell covariant derivative. Since the bosonic action used is a subset of the $d = 5, N = 2$ supergravity [8] it follows that the solutions preserve one of the two supersymmetries thus providing yet another example of the supersymmetry associated with self dual solutions of field equations [9]. The result of Ref.[1] allowed us to immedeately confirm [10] the conjecture [5, 6] concerning the gravitational stabilization of instantonic configurations in $SU(N), N \geq 3$ YMCS theory. The elegance and naturalness of the arguments of Ref.[1] lead one to expect that generalizations to other dimensions and types of solitons (strings, vortices etc.) should exist. In this letter we obtain solutions in 3 and $d = 4+p(p \geq 0)$ dimensions analogous to those of Ref.[1] in four and five dimensions. However the peculiarities of the spherically symmetric Greens function in two spatial dimensions (i.e the logarithm) lead to some peculiar properties for the solutions we generate. The metric decreases logarithmically with the distance from the vortex/string core and finally becomes *zero* at $r = r_H \sim l_S exp((\frac{l_S}{l_P})^{d-2})$ ($l_S, l_P$ are the vortex core size and the Planck length respectively). Strictly speaking, there is thus no region asymptotic to Minkowski space even though the vortex is localized on a scale $l_S$. However, in the intermediate asymptotic regime/region(IAR) $l_S << r << r_H$ the metric has only a logarithmic deviation from flatness with coefficient

$O((l_P/l_S)^{d-2})$. Similarly the ADM energy integral and the charge integral increase slowly with the size of the region of integration $R$ before finally diverging as $R \to r_H$. The Majumdar-Papapetrou electric field (in 4 dimensions the antisymmetric field strength) initially decreases marginally *slower* than that of a charged vortex/string (hence the prefix hyper in the title) and finally diverges at the signature changing horizon.

The action we shall treat is the sum of three pieces $S_{gr}, S_{SD}$ and $S_{CS}$ :

$$S_{gr} = -(16\pi G)^{-1} \int d^3xE(R + e^{2b\sigma}F^2 + 2(\partial\sigma)^2) \tag{1}$$

$$S_{SD} = -\int d^3xE(\frac{1}{4g^2}e^{2b_a\sigma}f^2 + e^{2b_\psi\sigma} \mid D\psi \mid^2 + \frac{g^2}{2}e^{2b_u\sigma}(\psi^\dagger\psi - v^2)^2) \tag{2}$$

$$S_{CS} = \int d^3x \quad \epsilon^{\mu\nu\lambda}A_\mu(\kappa_1\partial_\nu a_\lambda + \kappa_2\partial_\nu J_\lambda) \tag{3}$$

$F_{\mu\nu}, f_{\mu\nu}$ are field strengths of the MP ($A_\mu$) and NO ($a_\mu$) Abelian gauge potentials, $\psi$ the charged scalar field of the NO model, $\sigma$ the dilaton field and $J_\lambda = \frac{1}{2i}\psi^\dagger \overleftrightarrow{D}_\lambda\psi, (D_\lambda = \partial_\lambda\psi - ia_\lambda\psi)$ the NO current. Notice that the MP coupling has been equated to the gravitational one : $g_e^2 = 4\pi G$ The dilaton couplings $b, b_a, b_\psi, b_u$ will be chosen in the course of the calculation to allow an exact solution of the full theory given a flat space self dual NO vortex solution. The other possible Chern-Simons terms can be added to $S_{CS}$ without affecting our conclusions . Our conventions for gravitational quantities are those of [11], E is the determinant of the *dreibein* while $\epsilon^{\mu\nu\lambda}$ is the 3-d antisymmetric tensor density ($\epsilon^{012} = 1$).

To proceed we note that the MP form of the metric in $D + 1$ dimensions can be written in a form equivalent to that of [12] as ($i = 1, 2$) :

$$ds^2 = -\frac{1}{B^{D-2}}dt^2 + B(x^1, x^2)dx^idx^i \tag{4}$$

While this form of the metric can also be derived from considerations of the existence of a Killing spinor [1], one can view this ansatz simply as dictated by the need to ensure that the off diagonal terms of the Einstein tensor are automatically zero given that the spatial metric is conformal to the Euclidean one. In $2 + 1$ dimensions we therefore take the metric to be $Diag(-1, B, B)$. We also write $B = e^{2\phi}$.

The field equations are :

$$\tilde{G}_{\mu\nu} = G_{\mu\nu} + 8\pi G T_{\mu\nu}(A,\sigma) = -8\pi G T^{SD}_{\mu\nu}(a_\mu,\psi) \tag{5}$$

$$\partial_\mu(E e^{2b\sigma}F^{\mu\nu}) = -4\pi G \epsilon^{\nu\mu\lambda}\partial_\mu(\kappa_1 a_\lambda + \kappa_2 J_\lambda) \tag{6}$$

$$\begin{aligned}\partial_\mu(E g^{\mu\nu}\partial_\nu\sigma) - (b/2)e^{2b\sigma}EF^2 \;&=\; 8\pi G E((b_a/4g^2)e^{2b_a\sigma}f^2 + b_\psi e^{2b_\psi\sigma} \mid D\psi \mid^2 \\ &+\; (g^2/2)b_u e^{2b_u\sigma}(\psi^\dagger\psi - v^2)^2)\end{aligned} \tag{7}$$

$$\partial_\mu(e^{2b_a\sigma}E f^{\mu\nu}) = -2g^2 E e^{2b_\psi\sigma}J^\nu - g^2\epsilon^{\nu\mu\lambda}(\partial_\mu A_\lambda(\kappa_1 - \kappa_2\psi^\dagger\psi)) \tag{8}$$

$$D_\mu(e^{2b_\psi\sigma}E D^\mu\psi) = e^{2b_u\sigma}E\frac{\partial U}{\partial\psi^\dagger} + i\kappa_2\epsilon^{\nu\mu\lambda}\partial_\nu A_\mu D_\lambda\psi \tag{9}$$

where $U = (g^2/2)(\psi^\dagger\psi - v^2)^2$ is the potential.

When the metric is flat and $\sigma = 0, \kappa_i = 0$ the model reduces to the selfdual Abelian Higgs model whose field equations are solved and the energy minimized provided the fields are static, $a_0 = 0$, and the Bogomol'nyi equations:

$$(D_i \mp i\epsilon_{ij}D_j)\psi = 0 \tag{10}$$

$$f_{ij} = \pm\epsilon_{ij}g^2(\psi^\dagger\psi - v^2) \tag{11}$$

are satisfied. These first order equations may be decoupled to yield the well known vortex equation [13] :

$$\partial^2 ln\frac{\psi^\dagger\psi}{v^2} = 2g^2(\psi^\dagger\psi - v^2) + 4\pi\sum_k \mid n_k \mid \delta^{(2)}(\vec{r} - \vec{r}_k) \tag{12}$$

where $\{n_k, \vec{r}_k\}$ are the winding numbers and locations (positions of the zeros of $\psi$) of an ensemble of (anti)self dual vortices.

We now show that with a certain choice of dilaton weights every solution of the above self duality equations gives rise to an *exact, explicit* solution of the field equations of the full theory. We impose the relation :

$$e^{2b_\psi\sigma} = e^{2b_a\sigma}B^{-1} = Be^{2b_u\sigma} \tag{13}$$

which will be satisfied for a certain choice of weights provided $\sigma = \delta\phi$, $\delta$ a constant. Then it is easy to see, using the assumed flatspace self duality of the NO fields, that provided

$$\begin{aligned} A_0 &= \mp\frac{e^{2b_\psi \sigma}}{\kappa_2} \\ A_i &= 0 \end{aligned} \tag{14}$$

and $\kappa_1/\kappa_2 = v^2$ the field equations for the fields $a_\mu, \psi$ are satisfied. Furthermore it is easy to check that all spatial components of the Einstein tensor and the stress tensor of the matter sector vanish . Hence the spatial components of the Einstein equations are satisfied provided the spatial stress tensor of the fields $A_\mu, \sigma$ vanishes which requires that

$$\kappa_2^2 = b^2 \qquad\qquad b_\psi = -\frac{b}{2} \tag{15}$$

One finds that three remaining nontrivial field equations (i.e those for $G_{00}, A_0, \sigma$) reduce to the *single* flat space equation :

$$\begin{aligned} \partial^2 e^{2\phi} &= \pm(16\pi G)\epsilon_{ij}(\frac{v^2 f_{ij}}{2} + \partial_i J_j) \\ &\equiv -16\pi G(\frac{1}{4g^2} f_{ij}^2 + \mid D_i\psi \mid^2 + U(\psi)) \\ &\equiv (16\pi G)v^2(g^2(\psi^\dagger\psi - v^2) - \frac{1}{2}\partial^2(\frac{\psi^\dagger\psi}{v^2})) \end{aligned} \tag{16}$$

*if and only if* one identifies $\sigma = \phi, b = 2$ so that $b_\psi = -1, b_a = 0, b_U = -2$. The sign of $\kappa_2$ can be set without loss of generality so we put $\kappa_2 = -2$. Now, using the vortex equation, it immedeately follows that the regular solution of the above Poisson equation is:

$$e^{2\phi} = C + \mu(ln(\frac{\psi^\dagger\psi}{v^2}) - \frac{\psi^\dagger\psi}{v^2} - 2\sum_k \mid n_k \mid ln(v^2 \mid \vec{r} - \vec{r}_k \mid)) \tag{17}$$

where $C$ is an arbitrary constant and $\mu = 8\pi G v^2$. We can fix $C$ to be 1 by noting that as $G \to 0$ the gravitational, Maxwell and dilaton fields decouple and therefore we should recover the Minkowski metric.

In the region outside the core ($r >> \frac{1}{v^2}, \mid r_j \mid$) $B \to 1 - \mu(1 + 2Nln(v^2 r))$; ($\sum_k \mid n_k \mid = N$). Thus it follows that if we take G to be positive (see below) then the metric becomes degenerate at

$$r = r_H = l_S exp(\frac{1}{2N}(\frac{1}{\mu} - 1)) \approx l_S exp(\frac{1}{2\mu N}) \tag{18}$$

where $l_S = 1/v^2$ and $l_P = 16\pi G$ are the core size and Planck length respectively. Typically $l_S \geq 10^4 l_P$ (for GUT scale vortices) and can be as large as $10^{15} l_P$ for weak scale vortices. Thus the signature changing "horizon" at the fantastically large scale $r = r_H$ does not necessarily make our solutions (or rather the string solutions in 4 dimensions based on them) phenomenologically uninteresting since "frustrated structures " with strictly divergent energy (c.f global strings) can form during phase transitions due to the finite size of causally connected domains.

On the other hand , as is well known [14], the sign of Newton's constant in 3 dimensions is not fixed *a priori* since there is no static gravitational interaction to be made positive . In the present case, however, $G$ negative would also lead to a "wrong sign" for the Maxwell and dilaton field kinetic terms . The continuation of our solution beyond $r = r_H$ requires detailed analysis (since our solution relies for its consistency on the positivity of B) and will therefore be taken up elsewhere.

We next consider the behaviour of our solution in the "intermediate asymptotic region" $\frac{1}{v^2} << r \leq r_H$ . The ADM energy integral over the region $r \in [0, R]$; $\frac{1}{v^2} << R \leq r_H$ diverges logarithmically as $R \to r_H$ :

$$\begin{aligned} \mathcal{E} &= -\int_{r \leq R} d^2x T_0^0 B = -\frac{1}{8\pi G}\int d^2x \partial^2\phi \\ &= -\frac{rB'(r)}{8GB(r)}\Big|_0^R = \frac{2\pi v^2 N}{1 - 2\mu N ln(v^2 R)} \end{aligned} \tag{19}$$

where we have assumed that $R$ is much greater than the size of the region containing the different vortices. Note that the numerator is precisely the flatspace energy . Since $\mu = \frac{l_P}{l_S}$ is so small we see that for $R << r_H$ the energy increases very slowly with $R$. The weak divergence at $r = r_H$ could be cut off by a causal horizon, as can happen for global strings, or otherwise dynamically smoothed out.

Similarly we can calculate the electric field :

$$\begin{aligned} F^{0r} &= \frac{A_0'(r)}{B} \\ &= \pm \frac{\mu N}{r(1-2\mu N ln(v^2 r))^3} \end{aligned} \tag{20}$$

and the charge enclosed in radius $R$ :

$$\begin{aligned} Q &= \frac{1}{2}\int_{r=R} F^{\mu\nu} dS_{\mu\nu} \\ &= \pm \frac{2\pi\mu N}{(1-2\mu N ln(v^2 R))^2} \end{aligned} \tag{21}$$

Notice that the enclosed charge is proportional to the winding number as expected for a topological charge, but it also has the additional peculiar feature of an $N$ dependence in the denominator. Clearly both diverge at the signature changing horizon. Since, by Sylvester's theorem[11], the number of positive, negative and zero eigenvalues of the metric is preserved by a non-singular general coordinate transformation it follows that this horizon is not a coordinate artifact. It would be interesting to continue our solution beyond the horizon and study the consistency of the propagation of fluctuations on such a background [15].

String solutions in 4 dimensions are easily obtained from the above analysis by replacing the Maxwell potential $A_\mu$ by an antisymmetric gauge field $B_{\mu\nu}$ and appropriately modifying the terms involving the antisymmetric density. Thus one replaces the terms in the action containing $A_\mu$ by

$$\begin{aligned} S(B,...) &= -(16\pi G)^{-1}\int d^4x E e^{2b\sigma}\frac{1}{6}H_{\mu\nu\lambda}H^{\mu\nu\lambda} \\ &+ \frac{1}{2\sqrt{2}}\int d^4x \quad \epsilon^{\mu\sigma\nu\lambda}B_{\mu\sigma}(\kappa_1\partial_\nu a_\lambda + \kappa_2\partial_\nu J_\lambda) \end{aligned} \tag{22}$$

where $H_{\mu\nu\lambda} = \partial_{[\mu}B_{\nu\lambda]}$. Now if i) all fields are independent of $x^0, x^3$, ii) the only nonzero component of $B_{\mu\nu}$ is $B_{03} = \sqrt{2}A_0$ and the metric is $diag(-1, B, B, 1)$ then the new terms reduce to

$$
\begin{aligned}
S(B,...) &= (8\pi G)^{-1}\int d^4x E e^{2b\sigma}(\partial_i A_0)^2 \\
&+ \int d^4x \quad \epsilon^{ij} A_0(\kappa_1 \partial_i a_j + \kappa_2 \partial_i J_j)
\end{aligned} \tag{23}
$$

which is exactly the form of the action under our ansatz in the 3 dimensional case apart from the trivial extra integration which is balanced by the different dimensions of couplings and fields : $[B_{\mu\nu}] = [\kappa_2] = [g] = 0, [a_i] = [\psi] = 1$ etc. All our results for vortices thus carry over *mutatis mutandis* to the 4 dimensional case with the axionic charge replacing the electric one. Clearly by again raising the rank of the tensor potential we can embed our basic solution in any number of dimensions. Note that in d dimensions $\mu = \frac{l_P^{d-2}}{l_S^{d-2}}$ and so $r_H \approx v^{-\frac{2}{d-2}} e^{\frac{1}{2\mu N}}$ grows even faster with $1/l_P$. The pathologies of 3 dimensions and the special couplings of our 'dilaton' are responsible for the peculiarities of our solution. Nevertheless as an addition to the library of exact solutions of gravitationally coupled theories it merits further investigation. It motivates the search for further examples of the mechanism of Ref.[1] which involves as an essential part the aquisition of *topological electric fields* by magnetic configurations in the presence of CS type couplings [4, 5, 6, 1]. Moreover our solution may prove of use in studying the problem of the propagation of fluctuations on spacetimes with a signature change [15]. We have also found flatspace versions of these results (and of those for d=4+1 in Ref.[10]) by choosing suitable dilaton couplings.

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